

# Achieving Super-Resolution in Multi-Rate Sampling Systems via Efficient Semidefinite Programming

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**Abstract**—Super-resolution theory aims to estimate the discrete components lying in a continuous space that constitute a sparse signal with optimal precision. This work investigates the potential of recent super-resolution techniques for spectral estimation in multi-rate sampling systems. It shows that, under the existence of a common supporting grid, and under a minimal separation constraint, the frequencies of a spectrally sparse signal can be exactly jointly recovered from the output of a semidefinite program (SDP). The algorithmic complexity of this approach is discussed, and an equivalent SDP of minimal dimension is derived by extending the Gram parametrization properties of sparse trigonometric polynomials.

## I. INTRODUCTION

Compressed sensing techniques have proven to be of great interests for detecting, estimating and denoising sparse signals lying on discrete spaces. On the practical side, the applications of sparse modeling are many: single molecule imaging via fluorescence, blind source separation in speech processing, precise separation of multiple celestial bodies in astronomy, or super-resolution radar, are among those. However, the discrete gridding required by the compressed sensing framework weakens the recovery performances, and more precisely the *resolution*: the required minimal separation between two components of the sparse signal to be efficiently distinguished by an observation process.

In the recent years, a particular enthusiasm has been placed on solving sparse linear inverse problems over continuous spaces. This paradigm aims to recover the finite subset of components generating a signal, and lying in a continuous space, by discrete observations of this signal, distorted by a kernel function. Considering such approach raises new concerns, in particular, those problems are commonly infinitely ill-posed. This primordial issue has been addressed for the spikes model [1], [2], [3] via the mean of total-variation (or atomic) convex relaxation techniques, reducing the dimensionality on a dramatic manner. Later on, similar results have been derived for sparse signals lying on some known subspaces in [4], using particular kernel functions [5], or via incoherent multiple measurements in [6]. Generic performance in noise have been provided [7] and specific gradient search algorithms proposed in [8] to efficiently solve this category of problems.

For the spectral case, a complex time signal  $x$  is said to follow the  $s$ -spikes model if and only if it reads,

$$x(t) = \sum_{l=1}^s \alpha_l e^{i2\pi\xi_l t}, \quad \forall t \in \mathbb{R}, \quad (1)$$

where  $\xi = [\xi_1, \dots, \xi_s]^T \in \mathbb{R}^s$  is the vector containing the  $s$  spectral components generating the signal  $x$ , and  $\alpha \in \mathbb{C}^s$  the vector of their associated complex amplitudes. The frequency estimation problem is naturally defined as building a consistent estimator  $(\hat{s}, \hat{\xi}, \hat{\alpha})$  of the parameters  $(s, \xi, \alpha)$ , that are supposed to be unknown, by  $N \in \mathbb{N}$  discrete observations  $y \in \mathbb{C}^N$  of the time signal  $x$ .

This problem is obviously ill-posed, and since no assumption is a priori made on the number of frequencies  $\hat{s}$  to estimate, there are infinitely many triplet  $(\hat{s}, \hat{\xi}, \hat{\alpha})$  that are coherent with the observation vector  $y$ . In particular, the discrete Fourier transform of  $y$  forms a consistent spectral representation of  $x$  by  $N$  spectral spikes. Among all those estimators, the one considered to be optimal, in this context, will be the one returning the sparsest spectral distribution, i.e., the one achieving the smallest  $\hat{s}_0$ . The optimal spectral distribution  $\hat{x}_0$  can be written as the output of an optimization program taking the form,

$$\hat{x}_0 = \arg \min_{\hat{x} \in D_1} \|\hat{x}\|_0 \quad (2)$$

$$\text{subject to } y = \mathcal{F}(\hat{x}),$$

where  $\hat{x}$  is the spectral distribution of  $x$ ,  $\|\cdot\|_0$  represents the limit of the  $p$ -norm towards 0, counting the cardinality of the support.  $D_1$  denotes the space of absolutely integrable spectral distributions, and  $\mathcal{F}$  denotes a linear operator fully determined by the sampling process and linking the spectral domain to the measurements. Since this program is an NP-hard combinatorial problem, a common approach consists in relaxing the cardinality cost function into a minimization of the total-variation norm over the spectral domain, leading to the convex program,

$$\hat{x}_{\text{TV}} = \arg \min_{\hat{x} \in D_1} \|\hat{x}\|_{\text{TV}} \quad (3)$$

$$\text{subject to } y = \mathcal{F}(\hat{x}).$$

The previous works in the literature were mostly studying the regularly spaced observation model,  $y_k = x\left(\frac{k}{f}\right)$  for  $k \in \llbracket 0, N-1 \rrbracket$ . Under such observations, it has been shown that the relaxation proposed in (3) is exact in the sense that, under the minimal separation criterion over the normalized frequencies  $\Delta\nu = \min \{\text{frac}(\nu_i - \nu_j), i \neq j\} \geq \frac{4}{N-1}$  of the sparse spectral distribution  $\hat{x}$  to recover, the output of Programs (2) and (3) are identical. Additionally, Program (3) can be reformulated into a semidefinite program (SDP)

of dimension  $N + 1$ , where coefficients of the optimum define a trigonometric polynomial  $Q_*$  locating the frequencies of the original signal over the unit circle.  $Q_*$  takes modulus  $|Q_*(e^{i2\pi\nu})| = 1$  whenever  $2\pi f\nu = \xi_l$  and satisfies  $|Q_*(e^{i2\pi\nu})| < 1$  otherwise. It has been shown in [3] that this optimality still holds with high probability when extracting at random a small number of observations from  $y$  and discarding the rest of it. Other studies revealed that the spectral separation condition can be reduced [9], and that this model partially extends to multidimensional signals [10], [11].

In this work, our contribution is focused on extending the previous results on sparse frequency estimation to the framework of multi-rate sampling systems (MRSS): the observations  $y$  are gathered as the output of  $m$  different uniform samplers, working at different sampling rates, and potentially desynchronized (the samplers process the time signal  $x$  with some arbitrary delays). According to our knowledge, this approach is the first to extend super-resolution to such a generic measurement process. This model is of crucial importance, for instance, when seeking to achieve joint estimation of sparse signals in distributed sensor networks. Each node, with limited processing capabilities, samples at its own rate, a delayed version of a complex signal. Collected data are then sent and merged at a higher level processing unit, performing a global estimation of the spectral distribution on a joint manner. MRSS estimation is also a meaningful step towards a super-resolution theory from non-uniform sampling.

In Section II, we show in Proposition 3 that, under certain conditions on the rates and the delays between the samplers, the “total-variation” relaxation of the sparse recovery problem can take a polynomial form similar to the one described in the original paper [1]. We argue that the model benefits from the same performance guarantees, and from the optimality. We point out that this direct relaxation has an arbitrary high complexity, making it unsolvable by standard convex solvers. In Section III, a novel exact dimensionality reduction of the semidefinite form (8) is presented in Theorem 9 by extending the theory of Gram representation of trigonometric polynomials presented in [12] into the sparse case. We conclude that the dual of the main problem (5) can be reformulated in the compact SDP (10) whose dimension is equal to the number of observations.

## II. SUPER-RESOLVING MULTI-RATE SAMPLING SYSTEMS

### A. Observation model

An MRSS process on a continuous signal  $x$  is parametrized by a set  $\mathbb{A}$  of  $m$  distinct grids (or samplers)  $\mathcal{A}_j$ ,  $j \in \llbracket 1, m \rrbracket$ . Each grid is identified with a triplet  $\mathcal{A}_j = (f_j, \gamma_j, n_j)$ , where  $f_j \in \mathbb{R}^+$  is its sampling frequency,  $\gamma_j \in \mathbb{R}$  its delay (in sample unit), and  $n_j \in \mathbb{N}$  the number of measurements acquired by the grid. We assume those intrinsic characteristics to be known. The output  $y_j$  of the grid  $\mathcal{A}_j$  sampling a signal  $x$  following the sparse model described in (1) reads,

$$y_j[k] = \sum_{l=1}^s \alpha_l e^{i2\pi \frac{\xi_l}{f_j}(k-\gamma_j)}, \quad k \in \llbracket 0, n_j - 1 \rrbracket. \quad (4)$$

As explained above, the frequency estimation problem is formulated as finding the sparsest spectral density jointly matching the observation vectors  $y_j$ , for all  $j \in \llbracket 1, m \rrbracket$ . This problem takes the same form than the combinatorial minimization program (2), by specifying the equality constraint  $y = \mathcal{F}(\hat{x})$  as follows,

$$y_j = \mathcal{F}_j(\hat{x}), \quad \forall j \in \llbracket 1, m \rrbracket,$$

where  $\mathcal{F}_j$  is a linear operator denoting the effect of the spectral density on the samples uniformly acquired by the grid  $\mathcal{A}_j$ , and is characterized by,

$$\begin{aligned} \mathcal{F}_j : D_1 &\rightarrow \mathbb{C}^{n_j} \\ \hat{x} &\mapsto y_j : y_j[k] = \\ &\int_{\mathbb{R}} \hat{x}(\xi) e^{i2\pi \frac{\xi}{f_j}(k-\gamma_j)} d\xi, \quad \forall k \in \llbracket 0, n_j - 1 \rrbracket. \end{aligned}$$

### B. Convex relaxation

We recall that Program (2) is NP-hard in the general case, due to its combinatorial aspects. The relaxation described in (3) is introduced and takes the form,

$$\begin{aligned} \hat{x}_{\text{TV}} &= \arg \min_{\hat{x} \in D_1} \|\hat{x}\|_{\text{TV}} \\ \text{subject to} \quad &y_j = \mathcal{F}_j(\hat{x}), \quad \forall j \in \llbracket 1, m \rrbracket. \end{aligned} \quad (5)$$

Such transform has the advantage to turn the original infinite-combinatorial problem into a convex problem. However, for practical computation, convexity often is not enough in order to guarantee a successful resolution of a program. Indeed, the cost function of (5) takes values in  $D_1$ , a space having an uncountable dimension. Convex optimization theory ensures that this category of programs can be reformulated into semi-infinite programs [13]: a convex optimization program of a finite-dimensional cost function over an infinite-dimensional set of constraints, using the classic Lagrangian duality. In our settings, the Lagrange dual problem is,

$$\begin{aligned} c_* &= \arg \max \sum_{j=1}^m \Re(\langle y_j, c_j \rangle) \\ \text{subject to} \quad &\left\| \sum_{j=1}^m \mathcal{F}_j^*(c_j) \right\|_{\infty} \leq 1, \end{aligned} \quad (6)$$

where  $c = [c_1^T, \dots, c_m^T]^T$  is the dual variable, and  $\mathcal{F}_j^*$  denotes the adjoint of the operator  $\mathcal{F}_j$  for the Euclidean inner products. Since the original problem is only equally constrained, Slater's condition is automatically met, and strong duality holds. This implies that the optima of the primal problem (5) and its dual (6) are equal. Moreover this equality appends if and only if  $\hat{x}_{\text{TV}}$  is primal optimal, and  $c_*$  dual optimal [14].

Letting by  $\omega_j = \frac{2\pi\xi}{f_j}$  the normalized pulsation of array  $\mathcal{A}_j$ , the expression of the adjoint operator  $\mathcal{F}_j^*$  allows to reformulate the dual constraint into a boundedness constraint of a sum of

exponential polynomials of the form,

$$\begin{aligned}\mathcal{F}_j^*(c_j) &= \sum_{k=0}^{n_j-1} c_j[k] e^{-i(k-\gamma_j)\omega_j} \\ &= e^{i\gamma_j\omega_j} P_j(e^{-i\omega_j}),\end{aligned}$$

where  $P_j \in \mathbb{C}^{n_j-1}[X]$  is the dual complex polynomial related to array  $\mathcal{A}_j$ , and is defined by  $P_j(z) = \sum_{k=0}^{n_j-1} c_j[k] z^k$ .

### C. Common grid expansion

It has been shown in [1] that the sparse frequency recovery problem can take the form of a simple SDP when dealing with regularly spaced samples. However, those results cannot be transposed in the MRSS framework, since the dual constrained operator  $\sum_{j=1}^m \mathcal{F}_j^*(c_j)$  does not take a polynomial form. As an assumption to bridge this concern, the sampling process  $\mathbb{A}$  is supposed to admit a common supporting grid, ensuring that the observation samples can be uniformly aligned at a higher virtual rate. The notion of common supporting grid is defined bellow. Necessary and sufficient conditions in terms of the parameters of  $\mathbb{A}$  for its existence to hold are stated in Proposition 2.

**Definition 1.** A grid  $\mathcal{A}_\# = (f_\#, \gamma_\#, n_\#)$  is said to be a *common supporting grid* for a set of sampling grids  $\mathbb{A} = \{\mathcal{A}_j\}_{j \in \llbracket 1, m \rrbracket}$  if and only if the set of samples acquired by the MRSS induced by  $\mathbb{A}$  is a subset of the one acquired by  $\mathcal{A}_\#$ . In formal terms, the definition is equivalent to  $\left\{ \frac{1}{f_j} (k_j - \gamma_j), j \in \llbracket 1, m \rrbracket, k_j \in \llbracket 0, n_j - 1 \rrbracket \right\} \subseteq \left\{ \frac{1}{f_\#} (k_\# - \gamma_\#), k_\# \in \llbracket 0, n_\# - 1 \rrbracket \right\}$ . The set of common supporting grids of  $\mathbb{A}$  is denoted by  $\mathcal{C}(\mathbb{A})$ . Moreover, a common supporting grid  $\mathcal{A}_* = (f_*, \gamma_*, n_*)$  for  $\mathbb{A}$  is said to be *minimal* if and only if it satisfies the minimality condition,  $\forall \mathcal{A}_\# \in \mathcal{C}(\mathbb{A}), n_* \leq n_\#$ .

**Proposition 2.** Given a set of  $m$  observation grids  $\mathbb{A} = \{\mathcal{A}_j = (f_j, \gamma_j, n_j)\}_{j \in \llbracket 1, m \rrbracket}$ , a common supporting grid  $\mathcal{A}_\#$  exists if and only if there exist  $f_\# \in \mathbb{R}^+$ ,  $\gamma_\# \in \mathbb{R}$ , a set of  $m$  positive integers  $\{l_j\} \in \mathbb{N}^m$ , and a set of  $m$  integers  $\{a_j\} \in \mathbb{Z}^m$  satisfying  $f_\# = l_j f_j$  and  $\gamma_\# = l_j \gamma_j - a_j$  for all  $j \in \llbracket 1, m \rrbracket$ . Moreover a common grid  $\mathcal{A}_* = (f_*, \gamma_*, n_*)$  is minimal, if and only if,  $\gcd(\{a_j\}_{j \in \llbracket 1, m \rrbracket} \cup \{l_j\}_{j \in \llbracket 1, m \rrbracket}) = 1$ ,  $\gamma_* = \max_{j \in \llbracket 1, m \rrbracket} \{l_j \gamma_j\}$  and  $n_* = \max_{j \in \llbracket 1, m \rrbracket} \{l_j (n_j - 1) - a_j\}$ .

The proof of the above proposition is presented in [15]. In the following, we assume that  $\mathbb{A}$  satisfies the conditions of Proposition 2, and we denote its minimal common supporting grid by  $\mathcal{A}_* = (f_*, \gamma_*, n_*)$ . The next result shows that, under those circumstances, the dual inequality constraint in (6) takes a polynomial form.

**Proposition 3.** Consider the multi-rate sampling system induced by  $\mathbb{A} = \{\mathcal{A}_j\}_{j \in \llbracket 1, m \rrbracket}$ , if  $\mathcal{C}(\mathbb{A}) \neq \emptyset$  there exists a complex polynomial  $Q \in \mathbb{C}^{n_*-1}[X]$  such that  $\left\| \sum_{j=1}^m \mathcal{F}_j^*(c_j) \right\|_\infty = \|Q(e^{i\omega_*})\|_\infty$ .

*Proof:* The proof of this proposition is direct,

$$\begin{aligned}\sum_{j=1}^m \mathcal{F}_j^*(c_j) &= \sum_{j=1}^m e^{i\gamma_j\omega_j} P_j(e^{-i\omega_j}) \\ &= e^{i\gamma_*\omega_*} \sum_{j=1}^m e^{ia_j\omega_*} P_j(e^{-il_j\omega_*}),\end{aligned}$$

by replacing  $\omega_j$  by  $l_j\omega_*$  and  $l_j\gamma_j$  by  $\gamma_* + a_j$  in the second equality, where  $\{l_j\} \in \mathbb{N}^m$  and  $\{a_j\} \in \mathbb{Z}^m$  qualify the minimal common supporting grid  $\mathcal{A}_*$  of  $\mathbb{A}$ . It comes that,

$$\sum_{j=1}^m \mathcal{F}_j^*(c_j) = e^{i\gamma_*\omega_*} Q(e^{-i\omega_*}),$$

where  $Q(z) = \sum_{j=1}^m z^{-a_j} P_j(z^{l_j})$  is a well defined complex polynomial, since  $a_j \leq 0$  by assumption on the minimality of  $\mathcal{A}_*$ . Taking the infinite norm on both sides and noticing its invariance by  $\omega_* \leftarrow -\omega_*$  lead to the desired result. ■

Due to the upscaling effect created by the expansion of  $\mathbb{A}$  on a common grid  $\mathcal{A}_*$ , the resulting dual polynomial  $Q$  has a degree  $n_* - 1$  that can be potentially much higher than the initial degrees of the individual dual polynomials  $\{P_j\}_{j \in \llbracket 1, m \rrbracket}$ . This fact is illustrated by an example in the end of this section. However, it is easy to notice that  $Q$  is sparse, and that it can be decomposed into a sum over  $N_* \leq N = \sum_{j=1}^m n_j$  monomials. Let us denote by  $q \in \mathbb{C}^{n_*}$  the vector containing the coefficients of  $Q(z) = \sum_{k=0}^{n_*-1} q_k z^k$  and call by  $\mathcal{I} \subseteq \llbracket 0, n_* - 1 \rrbracket$ , the subset of cardinality  $N_*$  containing the powers of the supporting monomials. One can write the relation  $q = C_{\mathcal{I}} c$ , where  $c$  is the dual variable of Problem (6), for an orthogonal selection matrix  $C_{\mathcal{I}} \in [0, 1]^{n_* \times N_*}$  for the subset  $\mathcal{I}$ . The matrix  $C_{\mathcal{I}}$  can be directly inferred from the settings of  $\mathbb{A}$ .

Proposition 3 ensures that the dual constraint of the dual problem described in (6) is equivalent to restrict a complex polynomial to be bounded in modulus by one around the unit circle  $\mathbb{T}$ . We recall a result presented in [12] (Corollary 4.25) emerging from the Gram parametrization theory of complex polynomials which yields,

$$\|Q(e^{i\omega})\|_\infty \leq 1 \Leftrightarrow \exists H \text{ Hermitian s.t. } \begin{cases} \begin{bmatrix} H & q \\ q^H & 1 \end{bmatrix} \succeq 0 \\ \mathcal{T}_n^*(H) = e_1, \end{cases} \quad (7)$$

for any  $Q \in \mathbb{C}^{n-1}[X]$ , where  $\mathcal{T}_n^*$  is the adjoint to the canonical decomposition of Hermitian Toeplitz matrices of dimension  $n$   $\mathcal{T}_n$ , and is given by  $\mathcal{T}_n^*(H)[k] = \text{tr}(\Theta_k H)$ , for  $k \in \llbracket 0, n \rrbracket$ , where  $\Theta_k$  is the elementary Toeplitz matrix equals to 1 on the  $k^{\text{th}}$  lower diagonal and zero elsewhere, and where  $e_1$  is the first vector of the canonical basis of  $\mathbb{C}^n$ .

The semi-algebraic duality (7), combined with Proposition 3, allows to rewrite the infinite dimensional constraint of Program (6) into a positivity condition of an Hermitian matrix of dimension  $n_* + 1$  given by,

$$\begin{aligned}c_* &= \arg \max \Re(\langle y, c \rangle) \\ \text{subject to } & \begin{bmatrix} H & C_{\mathcal{I}} c \\ (C_{\mathcal{I}} c)^H & 1 \end{bmatrix} \succeq 0 \\ & \mathcal{T}_{n_*}^*(H) = e_1. \end{aligned} \quad (8)$$

The above problem is nothing but a particular case of the convex relations studied in [3]. This ensures that the optimum  $q_* = C_{\mathcal{I}} c_*$  induces a sparse complex polynomial  $Q_*$  that exactly locates the frequencies of  $x$  by solving  $|Q_*(e^{i\omega})| = 1$  around the unit circle  $\omega \in \mathbb{T}$ , as long as a sufficient minimal spectral separation discussed in [15] is respected.

Although semidefinite programs are theoretically solvable and certifiable, practical attempts to recover the frequencies of the time signal  $x$  via Program (8) might fail or return inaccurate results due to the high dimensionality of the constraints. This is the case in our settings, the square block matrix in (8) has a size of  $n_* + 1$ , which can be considerably higher than the effective dimension of the observations  $N_*$ , depending of the settings of the MRSS defined by  $\mathbb{A}$ . As for illustration purposes, suppose a delay-only MRSS, where  $\mathbb{A}$  is constituted of  $m$  grids given by  $\mathcal{A}_1 = (f, 0, n)$ ,  $\mathcal{A}_j = (f, -\frac{1}{b_j}, n)$  for all  $j \in \llbracket 2, m \rrbracket$ , and where the  $\{b_j\}_{j \in \llbracket 2, m \rrbracket}$  are jointly coprime. One has  $\mathcal{A}_* = ((\prod b_j) f, 0, (\prod b_j) n)$ , leading to a matrix constraint of asymptotic dimension  $\Omega(b^m n)$  for some constant  $b \in \mathbb{R}^+$ , while the essential dimension of the problem remains of order  $\mathcal{O}(mn)$ .

### III. EXACT DIMENSIONALITY REDUCTION

In this section, we show that the original dual problem described in (6) is equivalent to a similar SDP of size exactly equal to  $N_* + 1$ , which is optimal in those settings. To this end, we first need to recall some results about Gram parametrization of trigonometric polynomials.

#### A. Gram parametrization of trigonometric polynomials

For every non-zero complex number  $z \in \mathbb{C}^*$ , its  $n^{\text{th}}$  power vector is defined by  $\psi_n(z) = [1, z, \dots, z^n]^T$ . A complex trigonometric polynomial  $R \in \bar{\mathbb{C}}^n[X]$  of order  $\bar{n} = 2n + 1$  is a linear combination of complex monomials with positive and negative exponents absolutely bounded by  $n$ . Such polynomial  $R$  reads,

$$R(z) = \sum_{k=-n}^n r_k z^{-k}, \quad \forall z \in \mathbb{C}^*.$$

Each of such entities can be associated with a Gram set, as defined in Definition 4. Proposition 5 states that this duality holds via a simple linear relation with complex matrices.

**Definition 4.** A complex matrix  $G \in \mathbb{C}^{(n+1) \times (n+1)}$  is a *Gram matrix* associated with the trigonometric polynomial  $R$  if and only if,

$$R(z) = \psi(z^{-1})^T G \psi(z), \quad \forall z \in \mathbb{C}^*.$$

Such parametrization is, in general, not unique and we denote by  $\mathcal{G}(R)$  the set of matrices satisfying the above relation.  $\mathcal{G}(R)$  is called *Gram set* of  $R$ .

**Proposition 5.** For any complex trigonometric polynomial  $R$  of order  $\bar{n} = 2n + 1$ ,  $G \in \mathcal{G}(R)$  if and only if the relation,

$$\mathcal{T}_{\bar{n}}^*(G) = r$$

holds, where  $r \in \mathbb{C}^{\bar{n}}$  is the vector containing the coefficients of  $R$  indexed in  $\llbracket -n, n \rrbracket$ .

#### B. Compact representation of sparse polynomials

Up to here, the concept of Gram sets adapts to every complex trigonometric polynomial. If  $R$  is of order  $\bar{n}$ , it defines a set  $\mathcal{G}(R)$  of matrices in  $\mathbb{C}^{n \times n}$ . In our context,  $R$  has a sparse monomial support, and Gram representations with compact low-dimensional structures, reflecting this sparsity, are of crucial interest for the dimensionality reduction approach. Definition 6 introduces the notion of compact representations.

**Definition 6.** A complex trigonometric polynomial  $R$  of order  $\bar{n}$  is said to admit a *compact Gram representation* on a matrix  $M \in \mathbb{C}^{n \times m}$ ,  $m \leq n$ , if and only if there exists a matrix  $G_M \in \mathbb{C}^{m \times m}$  such that the relation,

$$\begin{aligned} R(z) &= \psi(z^{-1})^T M G_M M^T \psi(z) \\ &= \phi_M(z^{-1})^T G_M \phi_M(z), \quad \forall z \in \mathbb{C}^* \end{aligned}$$

holds, where  $\phi_M(z) = M^T \psi(z)$ . We denote by  $\mathcal{G}_M(R)$  the subset of complex matrices satisfying this property.

Although it can be difficult to characterize the set of polynomials admitting a compact representation on a given matrix  $M \in \mathbb{C}^{n \times m}$ , a simple criterion exists for the special case of selection matrices  $C_{\mathcal{I}}$ . This criterion is recalled from [12] in Proposition 7.

**Proposition 7.** A sparse trigonometric polynomial  $R \in \bar{\mathbb{C}}^n[X]$ , supported on  $\mathcal{J} \subseteq [-n, \dots, n]$ , admits a projected representation on a selection matrix  $C_{\mathcal{I}}$ ,  $\mathcal{I} \subseteq \llbracket 0, n \rrbracket$  if and only if  $\mathcal{J} \subseteq \mathcal{I} - \mathcal{I}$ .

#### C. Real bounded lemma for sparse polynomials

This part aims to demonstrate the novel Theorem 9, certifying that, when the polynomial  $Q$  is sparse, the condition  $\|Q(e^{i\omega})\|_{\infty} \leq 1$  is equivalent to the existence of a positive Hermitian matrix  $S$  (in a similar way as (7)), whose dimension is equal to  $N_* + 1$ , the essential dimension of Problem (5). We latter conclude on the existence of a compact SDP locating the spikes in  $\hat{x}$  with exact precision. The lemma bellow is first required for the demonstration of the main theorem.

**Lemma 8.** Let  $R \in \bar{\mathbb{C}}^n[X]$  and  $R' \in \bar{\mathbb{C}}^n[X]$  be two trigonometric polynomials with common monomial support on  $\mathcal{J} \subseteq \mathcal{I} - \mathcal{I} \subseteq [-n, \dots, n]$ . Let  $\mathcal{G}_{\mathcal{I}}(R)$  and  $\mathcal{G}_{\mathcal{I}}(R')$  be respectively the Gram compact sets of  $R$  and  $R'$  on the selection matrix  $C_{\mathcal{I}}$ . The inequality  $R'(e^{i\omega}) \leq R(e^{i\omega})$  holds for all  $\omega \in \mathbb{T}$  if and only if for every two Hermitian matrices  $G \in \mathcal{G}_{\mathcal{I}}(R)$  and  $G' \in \mathcal{G}_{\mathcal{I}}(R')$ , one has  $G' \preceq G$ .

*Proof:* By Proposition 7, the sets  $\mathcal{G}_{\mathcal{I}}(R)$  and  $\mathcal{G}_{\mathcal{I}}(R')$  are not empty. Thus, one can find two matrices  $G \in \mathcal{G}_{\mathcal{I}}(R)$  and  $G' \in \mathcal{G}_{\mathcal{I}}(R')$ . The inequality  $R'(e^{i\omega}) \leq R(e^{i\omega})$  holds for all  $\omega \in \mathbb{T}$  if and only if,  $0 \leq \phi_{C_{\mathcal{I}}}(e^{-i\omega})^T (G - G') \phi_{C_{\mathcal{I}}}(e^{i\omega})$  for all  $\omega \in \mathbb{T}$ . Since  $\phi_{C_{\mathcal{I}}}(e^{-i\omega}) = \overline{\phi_{C_{\mathcal{I}}}(e^{i\omega})}$  and by noticing that  $\{\phi_{C_{\mathcal{I}}}(\omega), \omega \in \mathbb{T}\}$  spans the whole space  $\mathbb{C}^{N_*}$ , we conclude that  $R'(e^{i\omega}) \leq R(e^{i\omega})$  for all  $\omega \in \mathbb{T}$  if and only if  $G' \preceq G$ . ■

**Theorem 9.** Let  $P$  and  $Q$  be two polynomials from  $\mathbb{C}^n[X]$  with common monomial support on  $\mathcal{I}$ . Define the trigonometric polynomial  $R(z) = P(z)P^*(z^{-1})$  for all  $z \in \mathbb{C}$ , and call by  $r \in \mathbb{C}^{n+1}$  the vector of its negative monomial coefficients such that  $R$  can be written under the form  $R(z) = r_0 + \sum_{k=1}^n (r_k z^{-k} + \bar{r}_k z^k)$ , for all  $z \in \mathbb{C}^*$ . Let by  $q \in \mathbb{C}^{n+1}$  the coefficients of  $Q$  and define by  $u \in \mathbb{C}^{|\mathcal{I}|}$  the vector satisfying  $q = C_{\mathcal{I}}u$ . Then the inequality,

$$|Q(e^{i\omega})| \leq |P(e^{i\omega})|, \quad \forall \omega \in \mathbb{T},$$

holds if and only if there exists a matrix  $S \in \mathbb{C}^{|\mathcal{I}| \times |\mathcal{I}|}$  satisfying the conditions,

$$\begin{cases} \begin{bmatrix} S & u \\ u^H & 1 \end{bmatrix} \succeq 0 \\ \mathcal{T}_{n^*}^*(C_{\mathcal{I}}SC_{\mathcal{I}}^H) = r. \end{cases} \quad (9)$$

*Proof:* The inequality  $|Q(e^{i\omega})| \leq |P(e^{i\omega})|$  is equivalent to  $|Q(e^{i\omega})|^2 \leq |P(e^{i\omega})|^2$  for all  $\omega \in \mathbb{T}$ . Denote by  $R$  and  $R'$  the two trigonometric polynomials  $R(e^{i\omega}) = |P(e^{i\omega})|^2$  and  $R'(e^{i\omega}) = |Q(e^{i\omega})|^2$ . It comes the equivalence with the inequality  $R'(e^{i\omega}) \leq R(e^{i\omega})$ , while  $R$  and  $R'$  are commonly supported by some subset  $\mathcal{J}$  satisfying  $\mathcal{J} \subseteq \mathcal{I} - \mathcal{I}$ .

Let  $q \in \mathbb{C}^{n+1}$  be the coefficients of the polynomial  $Q$ . Since  $R'$  is the square of  $Q$ , the rank one matrix  $qq^H$  belongs to  $\mathcal{G}(R')$ . Moreover,  $q$  is supported by the subset  $\mathcal{I}$ , if and only if there exists a  $u \in \mathbb{C}^{|\mathcal{I}|}$  such that  $q = C_{\mathcal{I}}u$ , and thus if and only if there exists a matrix  $uu^H \in \mathcal{G}_{\mathcal{I}}(R')$ .

By application of Lemma 8, an Hermitian matrix  $S \in \mathcal{G}_{\mathcal{I}}(R)$  satisfying  $S \succeq uu^H$  exists if and only if  $R(e^{i\omega}) \leq R'(e^{i\omega})$  for all  $\omega \in \mathbb{T}$ . We conclude by identification with a Schur complement that the block matrix inequality in (9) holds if and only if  $|Q(e^{i\omega})| \leq |P(e^{i\omega})|$ , for all  $\omega \in \mathbb{T}$ .

In addition, by Proposition 5,  $S \in \mathcal{G}_{\mathcal{I}}(R)$  is equivalent to  $\mathcal{T}_{n^*}^*(C_{\mathcal{I}}SC_{\mathcal{I}}^H) = r$ , which concludes the proof. ■

Applying Theorem 9 in the specific case where  $T(e^{i\omega}) = 1$ , for all  $\omega \in \mathbb{T}$ , the bounded polynomial constraint of Problem (6) verifies the semidefinite equivalence,

$$\begin{cases} \|Q(e^{i\omega})\|_{\infty} \leq 1 \\ q = C_{\mathcal{I}}c \end{cases} \Leftrightarrow \exists S \text{ Hermitian s.t. } \begin{cases} \begin{bmatrix} S & c \\ c^H & 1 \end{bmatrix} \succeq 0 \\ \mathcal{T}_{n^*}^*(C_{\mathcal{I}}SC_{\mathcal{I}}^H) = e_1. \end{cases}$$

where  $e_1$  is the first vector of the canonical basis of  $\mathbb{C}^{n^*}$ . Finally, we conclude on our main result, stating that Problem (6) is equivalent to the following reduced SDP,

$$\begin{aligned} c_* &= \arg \max \Re(\langle y, c \rangle) \\ \text{subject to } & \begin{bmatrix} S & c \\ c^H & 1 \end{bmatrix} \succeq 0 \\ & \mathcal{T}_{n^*}^*(C_{\mathcal{I}}SC_{\mathcal{I}}^H) = e_1. \end{aligned} \quad (10)$$

It is shown in [15] that due to the sparse structure of  $C_{\mathcal{I}}$ , the equality constraint in (10), involving vectors in  $\mathbb{C}^{n^*}$ , can be composed in  $o(N_*^2)$  independent linear forms, involving a total of  $\frac{N_*(N_*+1)}{2}$  variables, which do not degrade the computational complexity of Program (10). By equivalence, the

dual optima  $c_*$  returned by Problems (8) and (10) are similar. Consequently, the optimal polynomial  $Q_*(e^{i\omega})$ , locating the spikes in  $\hat{x}$ , can directly be recovered from the optimum  $c_*$  of the compact SDP (10) via the simple linear transform  $q_* = C_{\mathcal{I}}c_*$ .

#### IV. CONCLUSION

In this work, we extended the theory of super-resolution from discrete uniform samples to fit in the more generic framework of multi-rate sampling systems. We have shown that, under the existence of a virtual common supporting grid, one can build a dual polynomial locating with exact precision the frequencies, as long as a minimal separation criterion is met. The numerical complexity arising from this direct extension can be arbitrary high. We addressed this issue in the novel Theorem 9 by developing an equivalence between Hermitian matrices and bounded sparse polynomials over the unit circle. We have derived an equivalent SDP (10) of optimal dimension recovering the signal frequencies.

We reserve for a latter work a deeper exploration of the performances of this model, including a characterization of the resolution and spectral range benefits of MRSS, as well as an extension of this theory to non-uniform sampling systems, by removing the common grid hypothesis, that we believe to be artificial and unnecessary.

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